FS $\longrightarrow$ BFS

$$A\vec{x} = \vec{b}$$
$$\vec{x} \ge \vec{0}$$

FS and invertible

$$A\vec{x} = [B \mid R]\begin{bmatrix}\vec{x}_B \\ 0\end{bmatrix} = \vec{b}$$
$$\vec{x}_B \ge 0$$

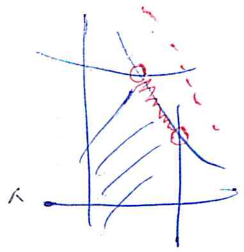
$$\boxed{\text{LPP} \quad \max \vec{c}^{\,T}\vec{x} \quad \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \ge 0 \end{cases}}$$

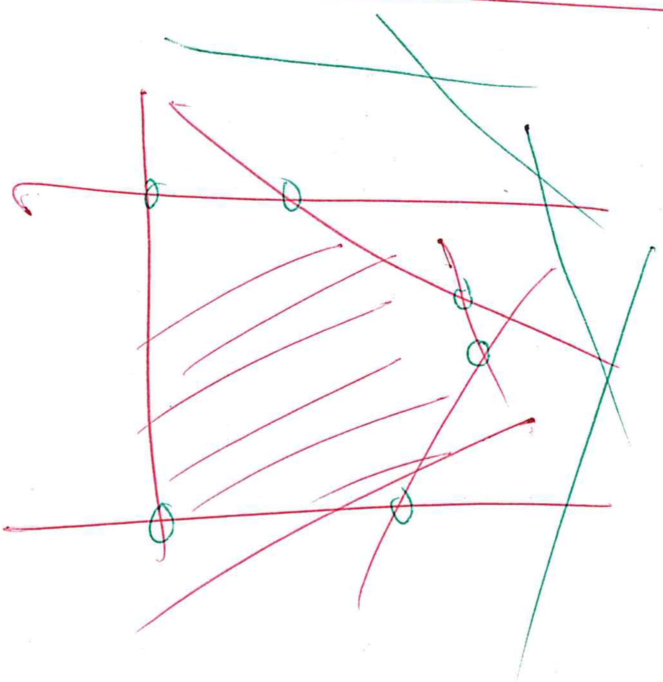
rank$(A) = m$

A m-by-n

BFS $\rightleftarrows$ Corner pt of FR

optimal solution occur at BFS





p2

Thm 2.1 BFS $\rightarrow$ Extreme pt of FR.

pf Let $\vec{x} \in BFS$ $\vec{x} = \begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix}$

(i) $[B \mid R] \begin{bmatrix} \vec{x}_B \\ \vec{0} \end{bmatrix} = \vec{b}$ B invertible,

(ii) $\vec{x}_B \geq \vec{0}$

pf by contradiction assume not an extreme pt.

$\exists$ $\vec{x}_1 \neq \vec{x}_2 \in FR$ s.t.

$$\vec{x} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2, \quad \lambda \in (0,1)$$

$\vec{x}_1, \vec{x}_2$ are FS : ① $[B \mid R] \underbrace{\begin{pmatrix} \vec{u}_1 \\ \vec{v}_1 \end{pmatrix}}_{\vec{x}_1} = \vec{b}$ ② $[B \mid R] \underbrace{\begin{pmatrix} \vec{u}_2 \\ \vec{v}_2 \end{pmatrix}}_{\vec{x}_2} = \vec{b}$ ①

$\vec{u}_1, \vec{v}_1 \geq \vec{0}$ $\vec{u}_2, \vec{v}_2 \geq \vec{0}$

$$\begin{pmatrix} \vec{x}_B \\ \vec{0} \end{pmatrix} = \lambda \begin{pmatrix} \vec{u}_1 \\ \vec{v}_1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} \vec{u}_2 \\ \vec{v}_2 \end{pmatrix} \quad \lambda \in (0,1)$$

$$\begin{cases} \vec{0} = \lambda \vec{v}_1 + (1-\lambda)\vec{v}_2 \quad \lambda \in (0,1) \\ \Rightarrow \vec{v}_1 = \vec{v}_2 = \vec{0}. \quad ② \end{cases}$$

② $\rightarrow$ ① $B\vec{u}_1 = \vec{b}$ $B\vec{u}_2 = \vec{b}$

$\Rightarrow$ $\vec{u}_1 = B^{-1}\vec{b} = \vec{u}_2$ ③

② & ③ $\Rightarrow$ $\vec{x}_1 = \vec{x}_2$ Contradiction

Thm 2.4 Extreme pt of FR $\Rightarrow$ BFS

pf: $\vec{x}_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ r can be equal to n

ex pt of FR. (FS)

$\begin{cases} A\vec{x}_0 = \vec{b} \\ \vec{x}_0 \geq 0 \end{cases} \Leftrightarrow \left\{ [\vec{a}_1, \vec{a}_2 \ldots \vec{a}_r, \vec{a}_{r+1} \ldots \vec{a}_n] \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{b} \right.$ ①

$x_1 \ldots, x_r \neq 0$

Case 1: ~~If~~ $\{\vec{a}_1, \ldots \vec{a}_r\}$ is li

Case 2 - - - - is ld

Case 1: $r \leq m = \text{rank}(A)$

~~Choose~~ $\exists$ another $m-r$ columns of A s.t.

$\{\vec{a}_1 \ldots, \vec{a}_r, \vec{a}_{r+1} \ldots, \vec{a}_m\}$ is li.

$$\Big[\underbrace{\vec{a}_1 \ldots \vec{a}_r \,\Big|\, \vec{a}_{r+1} \ldots \vec{a}_m}_{\text{invertible}} \Big| \ldots\ldots \Big] \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} r \\ \\ m \\ \\ n \end{matrix} = \vec{b}$$

$$[B \mid R] \begin{bmatrix} x_r \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ m \\ \\ n \end{matrix} = \vec{b} \quad \text{(degenerate)} \Rightarrow \text{BFS}$$

Case 2 $\{\vec{a}_1, \dots, \vec{a}_r\}$ l.d. (Can never happen)

Proof by Contradiction:

Suppose $\{\vec{a}_1, \dots \vec{a}_r\}$ l.d.

$\exists\ \alpha_1, \dots, \alpha_r$ not all zero s.t.

$$\alpha_1 \vec{a}_1 + \cdots + \alpha_r \vec{a}_r = \vec{0} \iff$$

$$[\vec{a}_1, \dots \vec{a}_r, \dots] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$\iff A\vec{\alpha} = \vec{0}$$

$$\varepsilon = \min_{\substack{1 \le i \le r \\ |\alpha_i| \ne 0}} \left\{ \frac{x_i}{|\alpha_i|} \right\} > 0$$

$$\frac{x_i}{|\alpha_i|} \ge \varepsilon, \quad \forall i\ |\alpha_i| \ne 0$$

$$\Rightarrow x_i \ge \varepsilon |\alpha_i| \quad \forall i$$

$$\vec{x}_\pm = \vec{x}_0 \pm \varepsilon \vec{\alpha}$$

$$\vec{x}_+ = \vec{x}_0 + \varepsilon \vec{\alpha}$$
$$\vec{x}_- = \vec{x}_0 - \varepsilon \vec{\alpha}$$

(i) $\vec{x}_0 = \frac{1}{2}\vec{x}_+ + \frac{1}{2}\vec{x}_-$

$\vec{x}_+ \in$ FR ?? $\quad \vec{x}_- \in$ FR ??

Cannot be an ext pt of FR

(ii) $\vec{x}_+, \vec{x}_-$ feasible: $\quad \vec{x}_+ \ge \vec{0}$, $\vec{x}_- \ge \vec{0}$

$\Rightarrow x_i \pm \varepsilon \alpha_i \ge 0 \quad \forall i$ #

(iii) $\vec{x}_+, \vec{x}_-$ solution $\quad A\vec{x}_+ = \vec{b}$?? $\quad A\vec{x}_- = \vec{b}$??

$$A\vec{x}_+ = A(\vec{x}_0 + \varepsilon \vec{\alpha})$$
$$= A\vec{x}_0 + \varepsilon A\vec{\alpha}$$
$$= \vec{b} + \vec{0} = \vec{b}$$

Thm 2.5 Optimal Solution of LPP occurs at ex pt of FR.

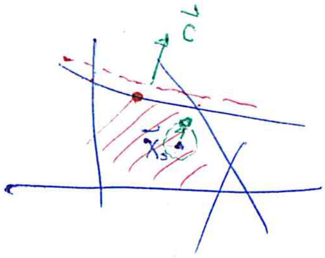
pf: $\vec{x}_0 \in FR$ (i) $A\vec{x}_0 = \vec{b}$

(ii) $\vec{x}_0 \geq \vec{0}$

(iii) $\vec{c}^T x_0 \geq \vec{c}^T \vec{x} \quad \forall \vec{x} \in FR$

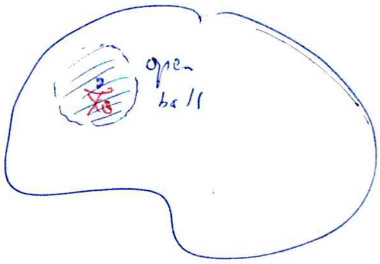
Step 1 $\vec{x}_0$ cannot be an interior pt

Step 2 $\vec{x}_0$ boundary pt $\longrightarrow$ $\exists$ corner pt that is optimal



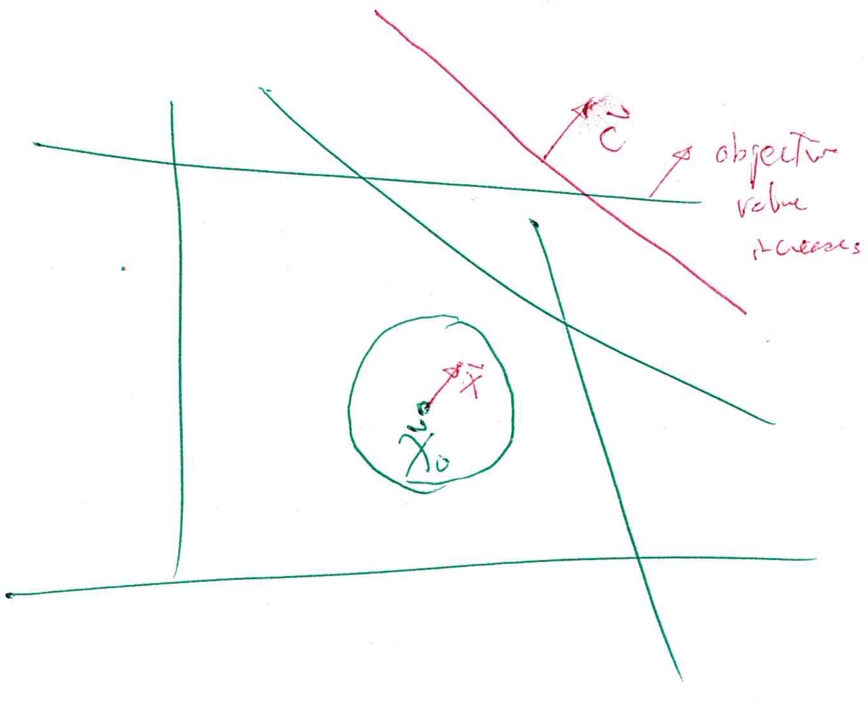
Step 1: $\vec{x}_0$ is interior pt

pf by contradict



$\exists\ B_\varepsilon(\vec{x}_0) \subseteq FR \qquad \varepsilon > 0$

$B_\varepsilon(\vec{x}_0) = \{ \vec{x} \mid |\vec{x} - \vec{x}_0| < \varepsilon \}$

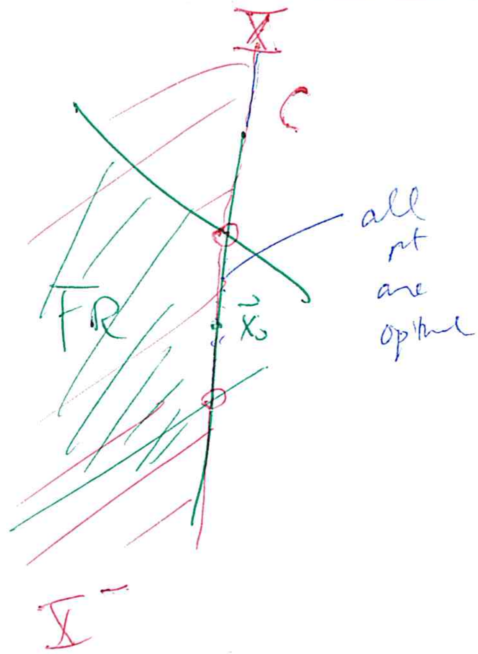


$$\vec{x} = \vec{x}_0 + \frac{\varepsilon}{2}\frac{\vec{c}}{|\vec{c}|} \in FR$$

$$\vec{c}^T\vec{x} = \underbrace{\vec{c}^T\vec{x}_0}_{=\text{ original optimal value}} + \underbrace{\frac{\varepsilon}{2}\frac{\vec{c}^T\vec{c}}{|\vec{c}|}}_{>0}$$

$\boxed{\text{a contradiction}}$

Step 2 $\quad \vec{x}_0$ is boundary pt



Define $X = \{\vec{x} \mid \vec{c}^T\vec{x} = \underbrace{\vec{c}^T\vec{x}_0}_{\text{optimal value}}\}$

(i) $\vec{x}_0 \in X$ supporting of $\vec{x}_0$

(ii) $X^- = \{\vec{x} \mid \vec{c}^T\vec{x} \le \vec{c}^T\vec{x}_0\}$

$\vec{x} \in FR \subseteq X^-$

$\Rightarrow$ X is a supporting hyperplane of FR at $\vec{x}_0$

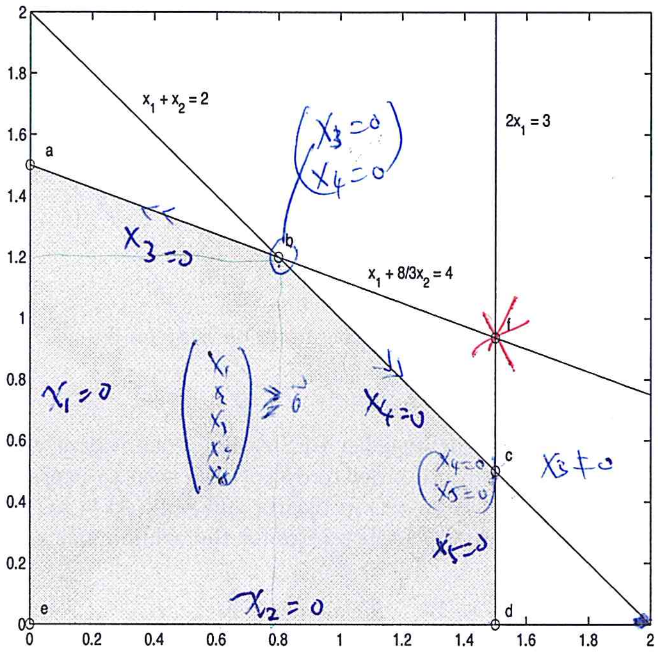
Thm 1.5 $\Rightarrow$ $\exists$ ex pt of FR on X.

$\Rightarrow$ that ex pt has to be optimal

($\because$ every pt on X is optimal)

#

p 7



$$\begin{bmatrix} 1 & 8/3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 8/3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

Figure 2.1. *There are 5 extreme points by inspection* $\{a, b, c, d, e\}$.

We get the point $[0, \frac{3}{2}, 0, \frac{1}{2}, 3]$. From Figure 2.3, we see that this point corresponds to the extreme point a of the convex polyhedron K defined by (2.19), (2.20), (2.21), (2.22). The equations $x_1 = 0$ and $x_3 = 0$ are called the *binding equations* of the extreme point a.

We note that not all basic solutions are feasible. In fact there is a maximum total of $\binom{5}{3} = \binom{5}{2} = 10$ basic solutions and here we have only five extreme points. The extreme points of the region K and their corresponding binding equations are given in the following table.

extreme point	a	b	c	d	e
set to zero	x_1	x_3	x_4	x_2	x_1
	x_3	x_4	x_5	x_5	x_2

If we set x_3 and x_5 to zero, the basic solution we get is $[\frac{3}{2}, \frac{15}{16}, 0, -\frac{7}{16}, 0]$. Hence it is not a basic *feasible* solution and therefore does not correspond to any one of the extreme point in K. In fact, it is given by the point f in the above figure.

2.3 Improving a Basic Feasible Solution

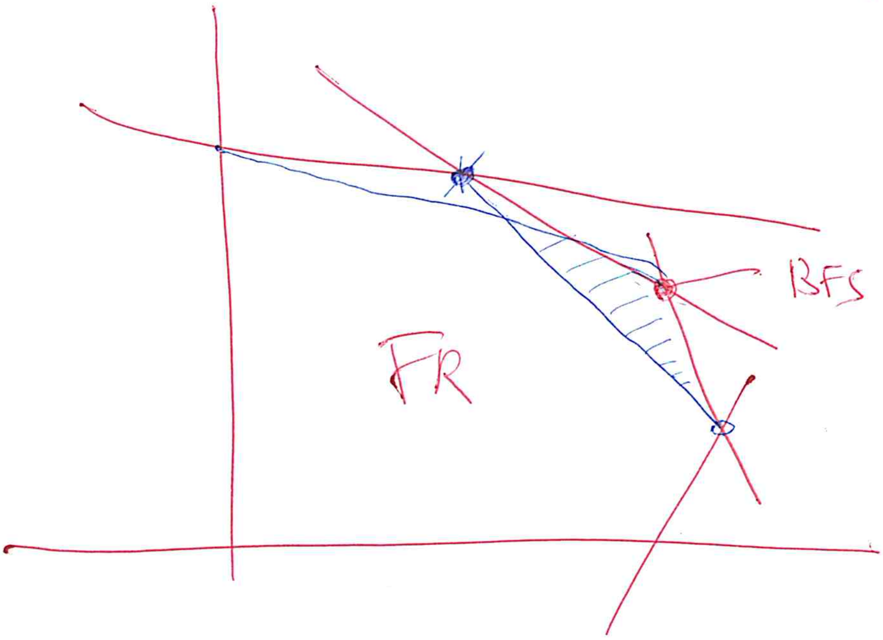
Let the LPP be

$$\max \quad z = \mathbf{c}^T\mathbf{x}$$

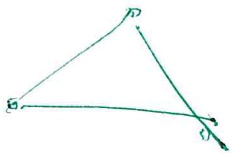
$$\text{subject to} \quad \begin{cases} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \, . \end{cases}$$

Here we assume that $\mathbf{b} \geq \mathbf{0}$ and $\text{rank}(A) = m$. Let the columns of A be given by $\mathbf{a}_j$, i.e. $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$. Let B be an $m \times m$ non-singular matrix whose columns are linearly independent

Simplex

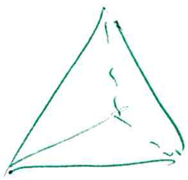


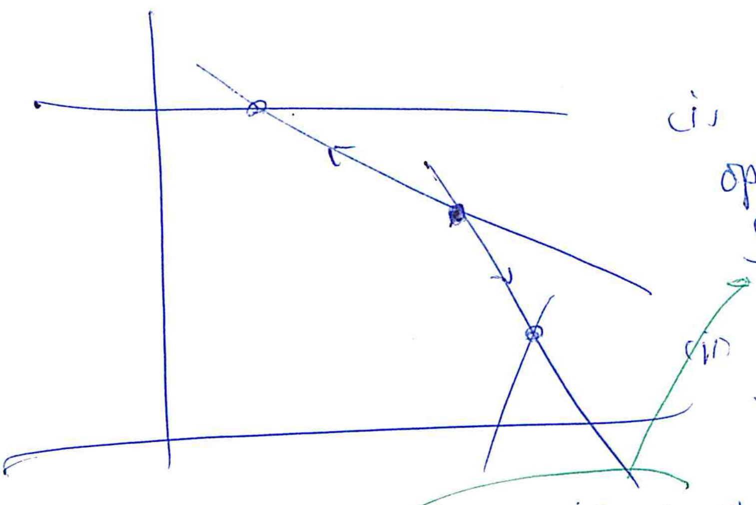
$\mathbb{R}^2$ 3 pt in $\mathbb{R}^2$ convex hull Simplex



n+1 vectors in $\mathbb{R}^n$

$\mathbb{R}^3$ 4 pts in $\mathbb{R}^3$





(i) optimality condition

(ii) feasibility condition

FS ↳ BFS

Thm 2.2

瞎子爬山法

USID: Math3210
PASS: hyperplane

p9

Chapter 0

The Simplex Method for a Two-variable Problem

0.1 Interpretation of the Graphical Method

To introduce the basic ideas of the simplex method, we will use an example with only two decision variables x and y. We can then see how both the graphical method and the simplex method works. Consider

$$\text{Max} \quad f(x,y) = 30x + 20y$$
$$\text{subject to} \begin{cases} x + y \le 50 \\ 40x + 60y \le 2400 \\ x, y \ge 0 \end{cases} \qquad (0.1)$$

FCF

The graph of the feasible region is given in Figure 0.1.

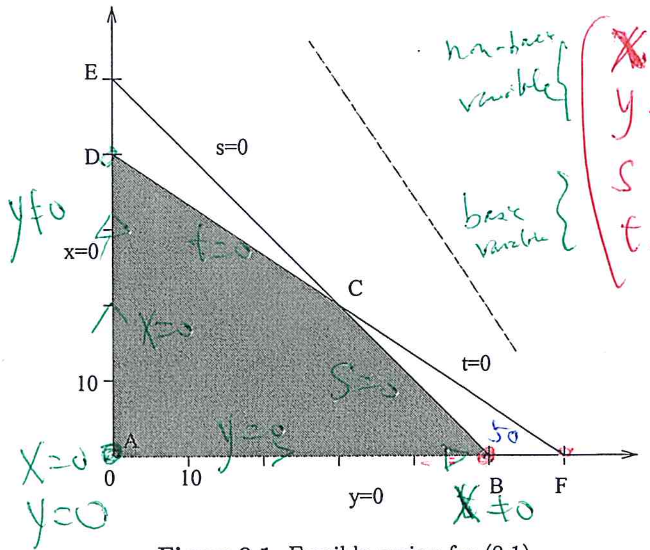


Figure 0.1. Feasible region for (0.1)

origin

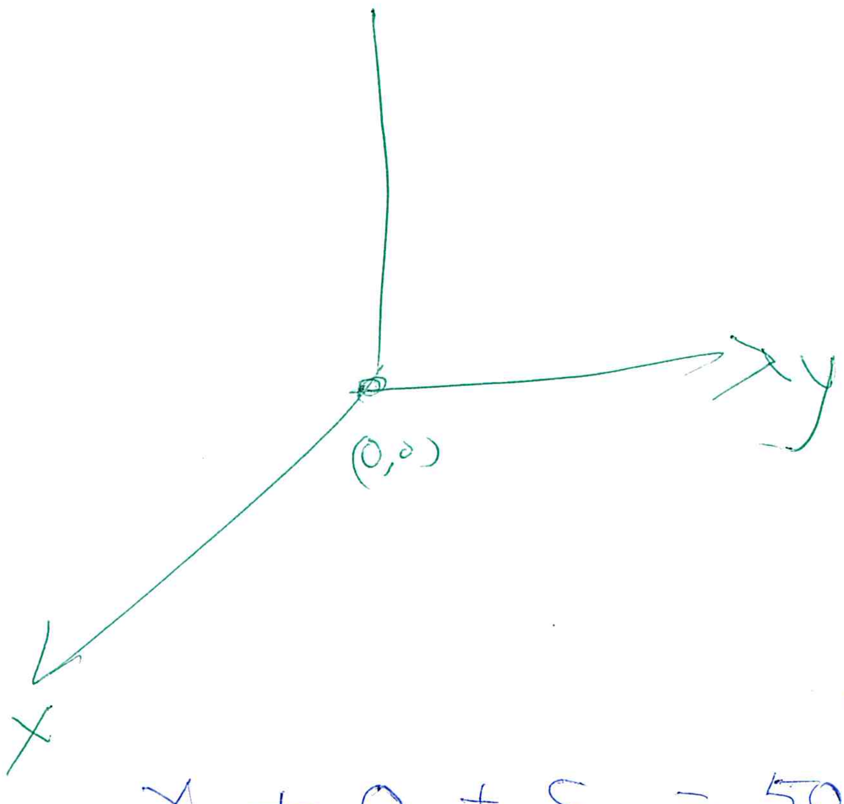
non-basic variable $\left\{ \begin{array}{l} x = 0 \\ y = 0 \end{array} \right.$

basic variable $\left\{ \begin{array}{l} s > 0 \\ t > 0 \end{array} \right.$

basic entering variable

$\begin{pmatrix} x > 0 \\ y = 0 \\ s \neq 0 \\ t \neq 0 \end{pmatrix}$

We note that the constraints are inequalities. Since inequalities are difficult to be handled by matrices, we first change them into equalities by adding two more variables



feasibility condt

$$\left.\begin{aligned} X + 0 + S &= 50 \Rightarrow \\ 40X \qquad + t &= 2400 \end{aligned}\right\} \boxed{\begin{aligned} x &= 50 \\ x &= 60 \end{aligned}}$$